

We will work over $\text{Sch}_{\mathbb{A}^1}$ unless stated otherwise.

Def. A morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of prestacks is representable by schemes if $\forall T \rightarrow Y$ w/ T scheme: $\mathcal{X} \times_Y T$ scheme. (resp. representable)
(resp. alg. space) $\rightarrow$ see def below

Def. $\mathcal{P}$ = property of morphisms of schemes stable under base change.
(resp. & satisfy (H))

A morphism of prestacks $\mathcal{X} \rightarrow \mathcal{Y}$ representable by schemes $\mathcal{X}' \rightarrow \mathcal{X} \rightarrow \mathcal{Y}$ has $\mathcal{P}$ (resp. representable) $\Leftrightarrow \mathcal{X} \rightarrow \mathcal{Y}$ has $\mathcal{P}$
 $\hookrightarrow \forall X' \rightarrow X$ surj étale:

has property $\mathcal{P}$ if $\forall T \rightarrow Y$ w/ T scheme: $X \times_Y T \rightarrow T$ has $\mathcal{P}$.
(resp. $\forall T \rightarrow Y$ w/ T scheme & étale presentation $U \rightarrow \mathcal{X} \times_Y T: U \rightarrow \mathcal{X} \times_Y T \rightarrow T$ has $\mathcal{P}$.)

Def. An algebraic space is a sheaf X s.t. $\exists$ scheme U & $U \rightarrow X$ representable by schemes, surj, étale.
 $\downarrow$
étale presentation of X

Def. A Deligne - Mumford stack is a stack $\mathcal{X}$ s.t. $\exists$ scheme U & $U \rightarrow \mathcal{X}$ (resp. algebraic stack)
representable, surj, étale.
 $\downarrow$
étale presentation of $\mathcal{X}$ (resp. smooth)
(resp. smooth)

Remark. (1) $\{\text{schemes}\} \subset \{\text{alg. spaces}\} \subset \{\text{DM stacks}\} \subset \{\text{alg. stacks}\}$

Inclusions are fully faithful & preserves fiber products (in the natural sense).

(2) There are different definitions of these objects in the literature, but they are all equivalent.

- Classical literature: additional conditions on representability of diagonal
- Olsson, Stacks Project: define them using Sch_{aff}

While Artin stack is usually synonymous with algebraic stack, it sometimes denotes an alg. stack locally of finite type over an excellent scheme w/ quasi-compact & separated diagonal.

(3) Roughly,

alg. space = sheaf that is a gluing of affine schemes on étale topology

DM stack = stack " " on " "

alg. stack = stack " " on smooth topology

Algebraicity of quotient stack

Theorem. $G \rightarrow S$ smooth affine group scheme acting on alg. space $U \rightarrow S$.

Then $U \rightarrow [U/G]$ is representable by schemes, surj, smooth, affine, and is a principal G -bundle.

Moreover, $[U/G]$ is an alg. stack.

Remark. To well-define the statement, we need the following:

- (1) We can define notions of group actions, G -equivariant morphisms, principal G -bundles, quotient prestacks / stacks on alg. spaces.

If G is affine, a principal G -bundle over a scheme is always a scheme. (Will need this in proof)

- (2) Let U an alg. space w/ a G -action & $\mathcal{X}$ stack.

A representable morphism $U \rightarrow \mathcal{X}$ is a principal G -bundle

if $\forall T \rightarrow \mathcal{X}$ w/ T scheme: $U \times_{\mathcal{X}} T \rightarrow T$ is a principal G -bundle.

 is an alg. space w/ G -action
induced by G -action on U

- (3) $\forall \mathcal{C}$ site, $S \in \text{Obj}(\mathcal{C})$: stack over $\mathcal{C}/_S =$ stack over $\mathcal{C}$
w/ morphism to S

pf. (of Theorem)

Let T an S -scheme, and $T \xrightarrow{\phi} [U/G]$ morphism.

Take $\begin{array}{ccc} P & \xrightarrow{\phi} & U \\ \pi \downarrow & & \\ T & & \end{array} \in \text{Obj} \left([U/G](T) \right)$ corresponding object to ϕ .

Then
$$\begin{array}{ccc} P & \xrightarrow{\pi} & T \\ \downarrow + & \square & \downarrow \phi \\ U & \longrightarrow & [U/G] \end{array}, \text{ and since } P \text{ is a scheme,}$$

we conclude $U \rightarrow [U/G]$ is representable by schemes.

Also $P \rightarrow T$ is a principal G -bundle, and so surj, smooth, affine.

$\Rightarrow U \rightarrow [U/G]$ has the same properties.

Lastly, to show $[U/G]$ is an algebraic stack, take $U' \rightarrow U$ étale presentation. Then $U' \rightarrow U \rightarrow [U/G]$ smooth presentation. $\blacksquare$

Cor. $G \rightarrow S$ smooth affine group scheme. The classifying stack $BG = [S/G]$ is an alg. stack.

Cor. $G \rightarrow S$ smooth affine group scheme acting on $U \rightarrow S$. ($\sigma: G \times_S U \rightarrow U$)
If G étale (over S) and the action on U is free (i.e. $G \times_S U \xrightarrow{(\sigma, pr_2)} U \times_S U$ mono),
 $[U/G]$ is an alg. space. (Hence we can denote it U/G)

In particular, if G is a finite group freely acting on an alg. space U , then $[U/G] = U/G$ is an alg. space.

pf. If group action is free, $[U/G]$ is a sheaf. (converse is also true)
 $G \rightarrow S$ étale $\Rightarrow$ any trivial G -bundle is étale
 $\Rightarrow$ any principal G -bundle is étale ($\because$ descent)

Since any base change of $U \rightarrow [U/G]$ by a scheme is a principal G -bundle (and hence étale), $U \rightarrow [U/G]$ is étale.

Taking $U' \rightarrow U$ étale presentation, $U' \rightarrow U \rightarrow [U/G]$ is an étale presentation. $\therefore [U/G]$ is an alg. space.

If G is a finite group, action by G is same as action by the group scheme $\bigsqcup_{g \in G} \text{Spec } \mathbb{Z}$ (over $\text{Spec } \mathbb{Z}$) which is étale. $\blacksquare$

Algebraicity of $\mathcal{M}_g$

Theorem. If $g \geq 2$, then $\mathcal{M}_g$ is an alg. stack over $\text{Spec } \mathbb{Z}$.

First we list some black boxes.

① (families of smooth curves)

$p: \mathcal{C} \rightarrow S$ family of smooth curves w/ genus $g \geq 2$.

(a) $w_{\mathcal{C}/S}^{\otimes 3}$ p -very ample

(b) $p_*(w_{\mathcal{C}/S}^{\otimes 2})$ locally free sheaf of rank $5(g-1)$

(c) adjunction map $p^*p_*(w_{\mathcal{C}/S}^{\otimes 2}) \rightarrow w_{\mathcal{C}/S}^{\otimes 3}$ is surj,

and the corresponding morphism $\mathcal{C} \rightarrow \mathbb{P}(p_*(w_{\mathcal{C}/S}^{\otimes 2}))$ is a closed immersion
(pullback of $\mathcal{O}(1)$ iso to $w_{\mathcal{C}/S}^{\otimes 2}$)

Also, the Hilbert polynomial of any fiber $\mathcal{C}_s \xrightarrow{\text{cl}} \mathbb{P}_{\kappa(s)}^{5g-6}$ ($\forall s \in S$)

$$\begin{aligned} \text{is } P(n) &:= \chi(\mathcal{O}_{\mathcal{C}_s}(n)) = \chi(w_{\mathcal{C}_s}^{\otimes 2n}) \underset{\text{(Riemann-Roch)}}{=} \deg(w_{\mathcal{C}_s}^{\otimes 2n}) + 1 - g \\ &= (6n-1)(g-1). \end{aligned}$$

② (Hilbert scheme)

$X \rightarrow S$ proj map of noetherian schemes. $\mathcal{O}_X(1)$ relatively ample. $P \in \mathbb{Q}[Z]$.

The functor $\text{Sch}/S \rightarrow \text{Set}$

$$(T \rightarrow S) \mapsto \left\{ \begin{array}{c} Z \xrightarrow{\text{cl}} X \times_S T \\ \text{flat} \searrow \downarrow \\ \text{\& finitely} \quad T \\ \text{presented} \end{array} \middle| \forall z \in T: \begin{array}{l} \text{Hilbert polynomial} \\ \text{of } Z_z \text{ is } P \end{array} \right\}$$

is represented by a scheme $\text{Hilb}^P(X/S)$ projective over S .

③ (When is an invertible sheaf a pullback of an invertible sheaf? / Prop A.8.16)

$f: X \rightarrow Y$ flat, proper morphism of noetherian schemes with geometrically integral fibers. Fix an invertible sheaf on X .

Then $\exists!$ closed subscheme $Z \hookrightarrow Y$ s.t. $T \rightarrow Y$ factors through Z

iff $L|_{X \times_Y T}$ is iso to a pullback of some invertible sheaf on T .

As sets, $Z = \{y \in Y \mid L_y \text{ is trivial}\}.$
 $\hookrightarrow$ invertible sheaf on X_y

Idea Observe family of smooth curves is "locally" a closed subscheme of $\mathbb{P}^{5-6}$ s.t. every fiber has Hilbert polynomial $P(n) = (6n-1)(g-1)$.

We aim to construct an iso from $\mathcal{M}_g$ to a quotient of a suitable subscheme of the Hilbert scheme $H := \text{Hilb}^P(\mathbb{P}^{5-6}/\mathbb{Z})$.

Also let $\mathcal{C} \xrightarrow{\text{cl}} \mathbb{P}^{5-6} \times H$ the universal family (etc corresponding to id_H on Hilbert functor) and $\pi: \mathcal{C} \rightarrow H$ projection.

We first prove the following Claim.

Claim. $\exists!$ locally closed subscheme $H' \subset H$ s.t.

automatic
since $\deg P = 1$

(a) $\forall h \in H': \mathcal{C}_h \rightarrow \text{Spec } \kappa(h)$ smooth & geom. connected curve
(so in particular, geom. integral)

(b) $\forall h \in H': \mathcal{C}_h \rightarrow \mathbb{P}^{5-6}_{\kappa(h)}$ is embedded via the complete linear system $|w_{\mathcal{C}_h/\kappa(h)}^{\otimes 3}|$
($\Leftrightarrow H^0(\mathbb{P}^{5-6}_{\kappa(h)}, \mathcal{O}(1)) \rightarrow H^0(\mathcal{C}_h, w_{\mathcal{C}_h/\kappa(h)}^{\otimes 3})$ iso)

(c) $w_{\mathcal{C}_h/H'}^{\otimes 3}$ and $\mathcal{O}_{\mathcal{C}_h}(1)$ differ by a pullback of an invertible sheaf from H'

Moreover, if $T \rightarrow H$ is a morphism s.t. (a), (b), (c) holds for $\mathcal{C}_T \rightarrow T$, then $T \rightarrow H$ factors through H' .

pf. We can construct H' by taking a sequence of open / closed subschemes one condition at a time.

(a) First, the smooth locus $\{h \in H \mid \mathcal{C}_h \text{ smooth}\} \subset H$ is open.

Consider the Stein factorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\pi} & H \\ \text{fibers} \searrow & & \nearrow \text{finite} \\ & \text{Spec } \pi_* \mathcal{O}_{\mathcal{C}} & \end{array}$$

= geom. conn

So $\{h \in H \mid \mathcal{C}_h \text{ geom. connected}\} = \{h \in H \mid \pi^*: \mathcal{O}_H \rightarrow \pi_* \mathcal{O}_{\mathcal{C}} \text{ iso on } h\}$
 $= H \setminus (\text{Supp}(\ker \pi^*) \cup \text{Supp}(\text{coker } \pi^*)) \subset H$ is open.

($\because \pi^*$ is a map of coherent $\mathcal{O}_H$ -modules)

$\therefore H_1 := \{h \in H \mid \mathcal{C}_h \text{ smooth \& geom. connected}\} \subset H$ is open.

(c) Apply ③ on $\pi_1: \mathbb{C}_{H_1} \rightarrow H_1$ and $\mathcal{O}_{\mathbb{C}_1}(1) \otimes \omega_{\mathbb{C}_{H_1}/H_1}^{\otimes -3}$ to get $H_2 \xrightarrow{\text{cl}} H_1$. (Recall π_1 has geom. integral fibers by (a))
 $\Rightarrow T \rightarrow H_1$ satisfies that $\omega_{\mathbb{C}_{H_1}/H_1}^{\otimes 2}|_T$ and $\mathcal{O}_{\mathbb{C}_T}(1)$ differ by a pullback of some invertible sheaf on T iff T factors through H_2 .
 In particular, $\forall h \in H_2: \omega_{\mathbb{C}_h/\pi(h)}^{\otimes 2} \cong \mathcal{O}_{\mathbb{C}_h}(1)$.

(b) Let $\pi_2: \mathbb{C}_{H_2} \rightarrow H_2$.

both locally free
of rank $5(g-1)$

Consider the canonical map $\alpha: H^0(\mathbb{P}_{\mathbb{Z}}^{5g-6}, \mathcal{O}(1)) \otimes_{\mathbb{Z}} \mathcal{O}_{H_2} \rightarrow \pi_{2,*} \mathcal{O}_{\mathbb{C}_{H_2}}(1)$.

The fiber of α over $h \in H_2$ is

$$\alpha_h: H^0(\mathbb{P}_{\pi(h)}^{5g-6}, \mathcal{O}(1)) \rightarrow H^0(\mathbb{C}_h, \mathcal{O}_{\mathbb{C}_h}(1)) \stackrel{(c)}{\cong} H^0(\mathbb{C}_h, \omega_{\mathbb{C}_h/\pi(h)}^{\otimes 2}).$$

$$\begin{aligned} \{h \in H_2 \mid (b) \text{ holds}\} &= \{h \in H_2 \mid \alpha_h \text{ is iso}\} \\ &= H_2 \setminus \text{Supp}(\text{coker}(\alpha)) \subset H_2 \text{ is open.} \end{aligned}$$

Take H' as this open subscheme.



pt. (of Theorem)

$\exists$ group action $\text{PGL}_{5g-5} \curvearrowright H$ defined as

$$\forall \begin{cases} g \in \text{Aut}(\mathbb{P}_S^{5g-5}) = \text{Hom}(S, \text{PGL}_{5g-5}) : \\ [D \xrightarrow{\text{cl}} \mathbb{P}_S^{5g-5}] \in H(S) \end{cases} :$$

$$g \cdot [D \xrightarrow{\text{cl}} \mathbb{P}_S^{5g-5}] = [g(D) \xrightarrow{\text{cl}} \mathbb{P}_S^{5g-5}].$$

$H' \subset H$ is loc. cl PGL_{5g-5} -invariant, so we have an induced group action

$$\text{PGL}_{5g-5} \curvearrowright H'. \text{ We aim to construct an iso } [H'/\text{PGL}_{5g-5}] \rightarrow \mathcal{M}_g$$

which will complete the proof.

First, construct $[H'/\text{PGL}_{5g-5}]^{\text{pre}} \rightarrow \mathcal{M}_g$.

$$\begin{array}{ccc} (S \rightarrow H') & \mapsto & (\mathbb{C} \rightarrow S) \\ \uparrow \text{corresponds with} & & \\ \mathbb{C} \xrightarrow{\text{cl}} \mathbb{P}_S^{5g-6} & & \downarrow \\ & \searrow & S \end{array}$$

I claim this map is fully faithful. To show this, it suffices to show maps between sets of automorphisms are bijections. (cf. Exercise 2.3.18 (a))
 Let $(S \rightarrow H') \mapsto (\mathcal{C} \xrightarrow{f} S)$ by the map.

$S \rightarrow H'$ corresponds with closed subscheme $\mathcal{C} \subset \mathbb{P}_S^{5g-6}$.

By condition (c), $\exists$ invertible sheaf M on S s.t. $\mathcal{O}_{\mathcal{C}}(1) \cong \omega_{\mathcal{C}/S}^{\otimes 2} \otimes p^*M$.

By condition (b), we have an isomorphism

$$H^0(\mathbb{P}_{\mathbb{Z}}^{5g-6}, \mathcal{O}(1)) \otimes_{\mathbb{Z}} \mathcal{O}_S \rightarrow p_* \mathcal{O}_{\mathcal{C}}(1) \cong p_*(\omega_{\mathcal{C}/S}^{\otimes 2} \otimes p^*M) \cong p_* \omega_{\mathcal{C}/S}^{\otimes 2} \otimes M.$$

Given an automorphism of $\mathcal{C} \rightarrow S$, it induces an automorphism of $p_* \omega_{\mathcal{C}/S}^{\otimes 2} \otimes M$, so by the iso above it induces an automorphism of $\mathbb{P}_S^{5g-6}$ that preserves $\mathcal{C}$. This is the unique automorphism of $S \rightarrow H'$ that maps to the given automorphism of $\mathcal{C} \rightarrow S$.

Since $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \rightarrow \mathcal{M}_g$ is fully faithful, the induced morphism

$[H'/\mathrm{PGL}_{5g-5}] \rightarrow \mathcal{M}_g$ is also fully faithful. (cf. Exercise 2.4.16)

To show $[H'/\mathrm{PGL}_{5g-5}] \rightarrow \mathcal{M}_g$ is an iso, it remains to show

it is essentially surjective. Given any object $\mathcal{C} \rightarrow S$ of $\mathcal{M}_g$, we need to show that $\exists \{S_i \rightarrow S\}$ étale cover s.t. $\mathcal{C}_{S_i} \rightarrow S_i$ is in the image of $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \rightarrow \mathcal{M}_g$. For this, just taking a Zariski open cover $\{S_i\}$ s.t. $(p_* \omega_{\mathcal{C}/S}^{\otimes 2})|_{S_i}$ is free works. $\square$

Remark. We can similarly show that $\mathrm{Coh}_{n,d}(\mathcal{C})$, $\mathrm{Bun}_{n,d}(\mathcal{C})$ are alg. stacks over $\mathrm{Spec} k$ by constructing isomorphisms to unions of quotient stacks, this time using Quot schemes.